Chapter ten Basic System Models

Objectives

10.1

Consider the following situation. A microprocessor switches on a motor. How will the rotation of the motor shaft vary with time? The speed will not immediately assume the full-speed value but will only attain that speed after some time. Consider another situation. A hydraulic system is used to open a valve which allows water into a tank to restore the water level to that required. How will the water level vary with time? The water level will not immediately assume the required level but will only attain that level after some time.

In order to understand the behaviour of systems, **mathematical models** are needed. These are simplified representations of certain aspects of a real system. Such a model is created using equations to describe the relationship between the input and output of a system and can then be used to enable predictions to be made of the behaviour of a system under specific conditions, e.g. the outputs for a given set of inputs, or the outputs if a particular parameter is changed. In devising a mathematical model of a system it is necessary to make assumptions and simplifications and a balance has to be chosen between simplicity of the model and the need for it to represent the actual real-world behaviour. For example, we might form a mathematical model for a spring by assuming that the extension x is proportional to the applied force F, i.e. $F = kx$. This simplified model might not accurately predict the behaviour of a real spring where the extension might not be precisely proportional to the force and where we cannot apply this model regardless of the size of the force, since large forces will permanently deform the spring and might even break it and this is not predicted by the simple model.

The basis for any mathematical model is provided by the fundamental physical laws that govern the behaviour of the system. In this chapter a range of systems will be considered, including mechanical, electrical, thermal and fluid examples.

Like a child building houses, cars, cranes, etc., from a number of basic building blocks, systems can be made up from a range of building blocks.

Each building block is considered to have a single property or function. Thus, to take a simple example, an electric circuit system may be made up from building blocks which represent the behaviour of resistors, capacitors and inductors. The resistor building block is assumed to have purely the property of resistance, the capacitor purely that of capacitance and the inductor purely that of inductance. By combining these building blocks in different ways, a variety of electric circuit systems can be built up and the overall input/output relationships obtained for the system by combining in an appropriate way the relationships for the building blocks. Thus a mathematical model for the system can be obtained. A system built up in this way is called a **lumped parameter** system. This is because each parameter, i.e. property or function, is considered independently.

There are similarities in the behaviour of building blocks used in mechanical, electrical, thermal and fluid systems. This chapter is about the basic building blocks and their combination to produce mathematical models for physical, real, systems. Chapter 11 looks at more complex models. It needs to be emphasised that such models are only aids in system design. Real systems often exhibit non-linear characteristics and can depart from the ideal models developed in these chapters. This matter is touched on in Chapter 11.

10.2

The models used to represent mechanical systems have the basic building blocks of springs, dashpots and masses. **Springs** represent the stiffness of a system, **dashpots** the forces opposing motion, i.e. frictional or damping effects, and **masses** the inertia or resistance to acceleration (Figure 10.1). The mechanical system does not have to be really made up of springs, dashpots and masses but have the properties of stiffness, damping and inertia. All these building blocks can be considered to have a force as an input and a displacement as an output.

The stiffness of a spring is described by the relationship between the forces F used to extend or compress a spring and the resulting extension or compression x (Figure 10.1(a)). In the case of a spring where the extension or compression is proportional to the applied forces, i.e. a linear spring,

$$F = kx$$

where k is a constant. The bigger the value of k, the greater the forces have to be to stretch or compress the spring and so the greater the stiffness. The

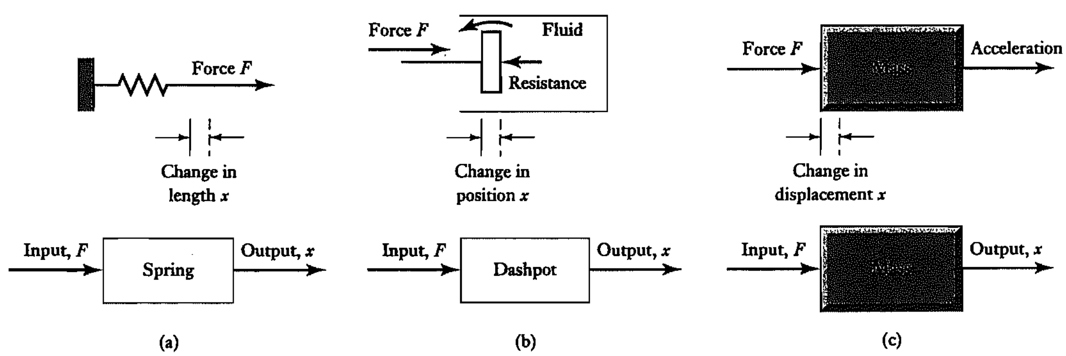


Figure 10.1 Mechanical systems: (a) spring, (b) dashpot, (c) mass.

object applying the force to stretch the spring is also acted on by a force, the force being that exerted by the stretched spring (Newton's third law). This force will be in the opposite direction and equal in size to the force used to stretch the spring, i.e. kx.

The dashpot building block represents the types of forces experienced when we endeavour to push an object through a fluid or move an object against frictional forces. The faster the object is pushed, the greater the opposing forces become. The dashpot which is used pictorially to represent these damping forces which slow down moving objects consists of a piston moving in a closed cylinder (Figure 10.1(b)). Movement of the piston requires the fluid on one side of the piston to flow through or past the piston. This flow produces a resistive force. In the ideal case, the damping or resistive force F is proportional to the velocity v of the piston. Thus

$$F = cv$$

where c is a constant. The larger the value of c, the greater the damping force at a particular velocity. Since velocity is the rate of change of displacement x of the piston, i.e. $v = \mathrm{d}x/\mathrm{d}t$, then

$$F = c\frac{\mathrm{d}x}{\mathrm{d}t}$$

Thus the relationship between the displacement x of the piston, i.e. the output, and the force as the input is a relationship depending on the rate of change of the output.

The mass building block (Figure 10.1(c)) exhibits the property that the bigger the mass, the greater the force required to give it a specific acceleration. The relationship between the force F and the acceleration a is (Newton's second law) $F = ma$, where the constant of proportionality between the force and the acceleration is the constant called the mass m. Acceleration is the rate of change of velocity, i.e. $\mathrm{d}v/\mathrm{d}t$, and velocity v is the rate of change of displacement x, i.e. $v = \mathrm{d}x/\mathrm{d}t$. Thus

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}(\mathrm{d}x/\mathrm{d}t)}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Energy is needed to stretch the spring, accelerate the mass and move the piston in the dashpot. However, in the case of the spring and the mass we can get the energy back but with the dashpot we cannot. The spring when stretched stores energy, the energy being released when the spring springs back to its original length. The energy stored when there is an extension x is $\frac{1}{2}kx^2$. Since $F = kx$ this can be written as

$$E = \frac{1}{2}\frac{F^2}{k}$$

There is also energy stored in the mass when it is moving with a velocity v, the energy being referred to as kinetic energy, and released when it stops moving:

$$E = \frac{1}{2}mv^2$$

However, there is no energy stored in the dashpot. It does not return to its original position when there is no force input. The dashpot dissipates energy rather than storing it, the power P dissipated depending on the velocity v and being given by

$$P = cv^2$$

10.2.1 Rotational systems

The spring, dashpot and mass are the basic building blocks for mechanical systems where forces and straight line displacements are involved without any rotation. If there is rotation then the equivalent three building blocks are a **torsional spring**, a **rotary damper** and the **moment of inertia**, i.e. the inertia of a rotating mass. With such building blocks the inputs are torque and the outputs angle rotated. With a torsional spring the angle θ rotated is proportional to the torque T. Hence

$$T = k\theta$$

With the rotary damper a disc is rotated in a fluid and the resistive torque T is proportional to the angular velocity ω, and since angular velocity is the rate at which angle changes, i.e. $\mathrm{d}\theta/\mathrm{d}t$,

$$T = c\omega = c\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

The moment of inertia building block has the property that the greater the moment of inertia I, the greater the torque needed to produce an angular acceleration α:

$$T = I\alpha$$

Thus, since angular acceleration is the rate of change of angular velocity, i.e. $\mathrm{d}\omega/\mathrm{d}t$, and angular velocity is the rate of change of angular displacement, then

$$T = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\frac{\mathrm{d}(\mathrm{d}\theta/\mathrm{d}t)}{\mathrm{d}t} = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

The torsional spring and the rotating mass store energy; the rotary damper just dissipates energy. The energy stored by a torsional spring when twisted through an angle θ is $\frac{1}{2}k\theta^2$ and since $T = k\theta$ this can be written as

$$E = \frac{1}{2}\frac{T^2}{k}$$

The energy stored by a mass rotating with an angular velocity ω is the kinetic energy E, where

$$E = \frac{1}{2}I\omega^2$$

The power P dissipated by the rotatory damper when rotating with an angular velocity ω is

$$P = c\omega^2$$

Table 10.1 summarises the equations defining the characteristics of the mechanical building blocks when there is, in the case of straight line

Table 10.1 Mechanical building blocks.

Building block	Describing equation	Energy stored or power dissipated
Translational		
Spring	$F = kx$	$E = \frac{1}{2}\frac{F^2}{k}$
Dashpot	$F = c\frac{\mathrm{d}x}{\mathrm{d}t} = cv$	$P = cv^2$
Mass	$F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\frac{\mathrm{d}v}{\mathrm{d}t}$	$E = \frac{1}{2}mv^2$
Rotational		
Spring	$T = k\theta$	$E = \frac{1}{2}\frac{T^2}{k}$
Rotational damper	$T = c\frac{\mathrm{d}\theta}{\mathrm{d}t} = c\omega$	$P = c\omega^2$
Moment of inertia	$T = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = I\frac{\mathrm{d}\omega}{\mathrm{d}t}$	$E = \frac{1}{2}I\omega^2$

displacements (termed translational), a force input F and a displacement x output and, in the case of rotation, a torque T and angular displacement θ.

10.2 Building up a mechanical system

Many systems can be considered to be essentially a mass, a spring and dashpot combined in the way shown in Figure 10.2(a) and having an input of a force F and an output of displacement x (Figure 10.2(b)). To evaluate the relationship between the force and displacement for the system, the procedure to be adopted is to consider just one mass, and just the forces acting on that body. A diagram of the mass and just the forces acting on it is called a **free-body diagram** (Figure 10.2(c)).

When several forces act concurrently on a body, their single equivalent resultant can be found by vector addition. If the forces are all acting along the same line or parallel lines, this means that the resultant or net force acting on the block is the algebraic sum. Thus for the mass in Figure 10.2(c), if we consider just the forces acting on that block then the net force applied to

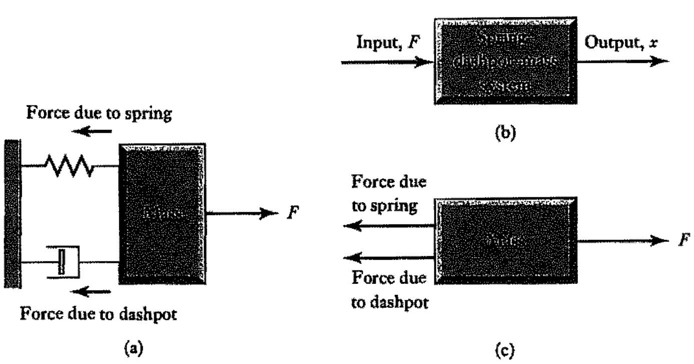


Figure 10.2 (a) Spring–dashpot–mass, (b) system, (c) free-body diagram.

the mass is the applied force F minus the force resulting from the stretching or compressing of the spring and minus the force from the damper. Thus

$$\text{net force applied to mass } m = F - kx - cv$$

where v is the velocity with which the piston in the dashpot, and hence the mass, is moving. This net force is the force applied to the mass to cause it to accelerate. Thus

$$\text{net force applied to mass} = ma$$

Hence

$$F - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

or, when rearranged,

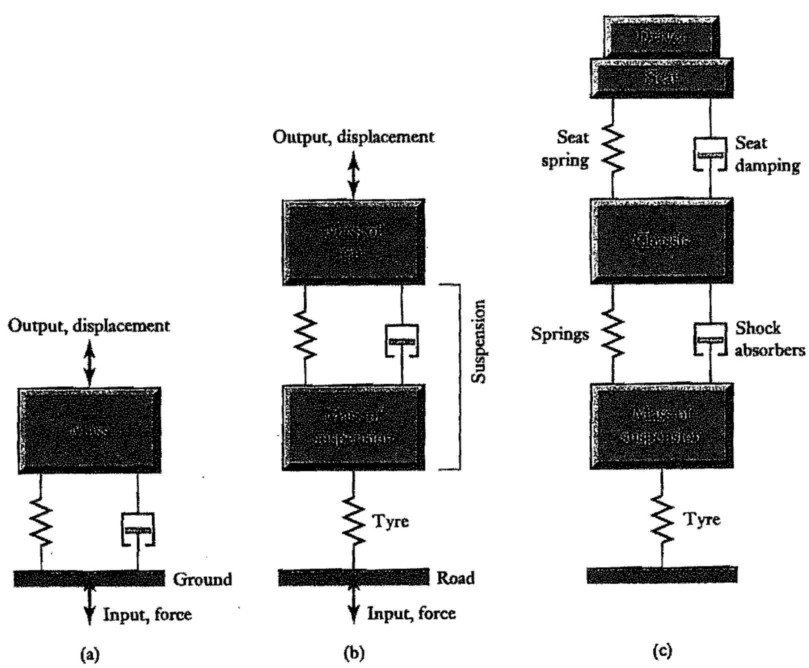
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

This equation, called a **differential equation**, describes the relationship between the input of force F to the system and the output of displacement x. Because of the d^2x/dt^2 term, it is a second-order differential equation; a first-order differential equation would only have dx/dt.

There are many systems which can be built up from suitable combinations of the spring, dashpot and mass building blocks. Figure 10.3 illustrates some.

Figure 10.3(a) shows the model for a machine mounted on the ground and could be used as a basis for studying the effects of ground disturbances on the displacements of a machine bed. Figure 10.3(b) shows a model for the wheel and its suspension for a car or truck and can be used for the study of the

Figure 10.3 Model for (a) a machine mounted on the ground, (b) the chassis of a car as a result of a wheel moving along a road, (c) the driver of a car as it is driven along a road.



behaviour that could be expected of the vehicle when driven over a rough road and hence as a basis for the design of the vehicle suspension. Figure 10.3(c) shows how this model can be used as part of a larger model to predict how the driver might feel when driven along a road. The procedure to be adopted for the analysis of such models is just the same as outlined above for the simple spring–dashpot–mass model. A free-body diagram is drawn for each mass in the system, such diagrams showing each mass independently and just the forces acting on it. Then for each mass the resultant of the forces acting on it is equated to the product of the mass and the acceleration of the mass.

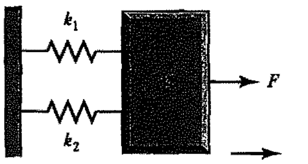


Figure 10.4 Example.

To illustrate the above, consider the derivation of the differential equation describing the relationship between the input of the force F and the output of displacement x for the system shown in Figure 10.4.

The net force applied to the mass is F minus the resisting forces exerted by each of the springs. Since these are k_1x and k_2x, then

$$\text{net force} = F - k_1x - k_2x$$

Since the net force causes the mass to accelerate, then

$$\text{net force} = m\frac{d^2x}{dt^2}$$

Hence

$$m\frac{d^2x}{dt^2} + (k_1 + k_2)x = F$$

The procedure for obtaining the differential equation relating the inputs and outputs for a mechanical system consisting of a number of components can be summarised as:

1 Isolate the various components in the system and draw free-body diagrams for each.
2 Hence, with the forces identified for a component, write the modelling equation for it.
3 Combine the equations for the various system components to obtain the system differential equation.

As an illustration, consider the derivation of the differential equation describing the motion of the mass m_1 in Figure 10.5(a) when a force F is applied. Consider the free-body diagrams (Figure 10.5(b)). For mass m_2 these are the force F and the force exerted by the upper spring. The force exerted by the upper spring is due to its being stretched by $(x_2 - x_3)$ and so is $k_2(x_3 - x_2)$. Thus the net force acting on the mass is

$$\text{net force} = F - k_2(x_3 - x_2)$$

This force will cause the mass to accelerate and so

$$F - k_2(x_3 - x_2) = m_2\frac{d^2x_3}{dt}$$

For the free-body diagram for mass m_1, the force exerted by the upper spring is $k_2(x_3 - x_2)$ and that by the lower spring is $k_1(x_1 - x_2)$. Thus the net force acting on the mass is

$$\text{net force} = k_1(x_2 - x_1) - k_2(x_3 - x_2)$$

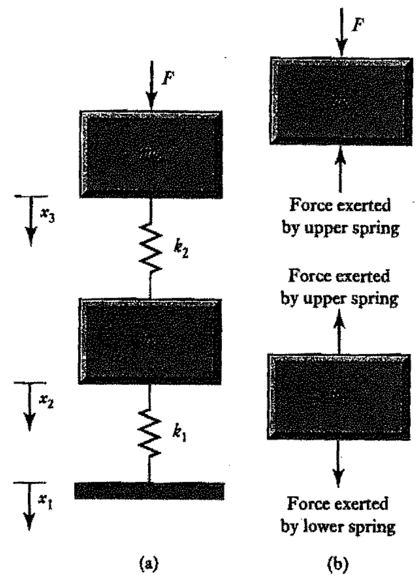


Figure 10.5 Mass–spring system.

This force will cause the mass to accelerate and so

$$k_1(x_2 - x_1) - k_2(x_3 - x_2) = m_1\frac{d^2x_2}{dt}$$

We thus have two simultaneous second-order differential equations to describe the behaviours of the system.

Similar models can be constructed for rotating systems. To evaluate the relationship between the torque and angular displacement for the system the procedure to be adopted is to consider just one rotational mass block, and just the torques acting on that body. When several torques act on a body simultaneously, their single equivalent resultant can be found by addition in which the direction of the torques is taken into account. Thus a system involving a torque being used to rotate a mass on the end of a shaft (Figure 10.6(a)) can be considered to be represented by the rotational building blocks shown in Figure 10.6(b). This is a comparable situation with that analysed above (Figure 10.2) for linear displacements and yields a similar equation

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = T$$

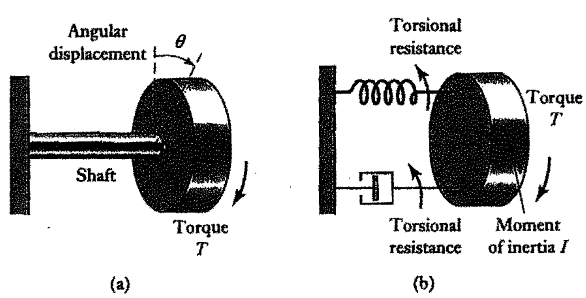


Figure 10.6 Rotating a mass on the end of a shaft: (a) physical situation, (b) building block model.

10.3

The basic building blocks of electrical systems are inductors, capacitors and resistors (Figure 10.7).

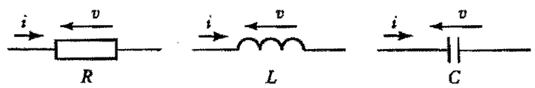


Figure 10.7 Electrical building blocks.

For an **inductor** the potential difference v across it at any instant depends on the rate of change of current $(\mathrm{d}i/\mathrm{d}t)$ through it:

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

where L is the inductance. The direction of the potential difference is in the opposite direction to the potential difference used to drive the current through the inductor, hence the term back e.m.f. The equation can be rearranged to give

$$i = \frac{1}{L}\int v\,\mathrm{d}t$$

For a **capacitor**, the potential difference across it depends on the charge q on the capacitor plates at the instant concerned:

$$v = \frac{q}{C}$$

where C is the capacitance. Since the current i to or from the capacitor is the rate at which charge moves to or from the capacitor plates, i.e. $i = \mathrm{d}q/\mathrm{d}t$, then the total charge q on the plates is given by

$$q = \int i\,\mathrm{d}t$$

and so

$$v = \frac{1}{C}\int i\,\mathrm{d}t$$

Alternatively, since $v = q/C$ then

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{C}\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{C}i$$

and so

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}$$

For a **resistor**, the potential difference v across it at any instant depends on the current i through it

$$v = Ri$$

where R is the resistance.

Both the inductor and capacitor store energy which can then be released at a later time. A resistor does not store energy but just dissipates it. The energy stored by an inductor when there is a current i is

$$E = \frac{1}{2}Li^2$$

The energy stored by a capacitor when there is a potential difference v across it is

$$E = \frac{1}{2}Cv^2$$

The power P dissipated by a resistor when there is a potential difference v across it is

$$P = iv = \frac{v^2}{R}$$

Table 10.2 summarises the equations defining the characteristics of the electrical building blocks when the input is current and the output is potential difference. Compare them with the equations given in Table 10.1 for the mechanical system building blocks.

Table 10.2 Electrical building blocks.

Building block	Describing equation	Energy stored or power dissipated
Inductor	$i = \frac{1}{L}\int v\,dt$	$E = \frac{1}{2}Li^2$
	$v = L\frac{di}{dt}$	
Capacitor	$i = C\frac{dv}{dt}$	$E = \frac{1}{2}Cv^2$
Resistor	$i = \frac{v}{R}$	$P = \frac{v^2}{R}$

10.3.1 Building up a model for an electrical system

The equations describing how the electrical building blocks can be combined are **Kirchhoff 's laws.** These can be expressed as:

Law 1: the total current flowing towards a junction is equal to the total current flowing from that junction, i.e. the algebraic sum of the currents at the junction is zero.

Law 2: in a closed circuit or loop, the algebraic sum of the potential differences across each part of the circuit is equal to the applied e.m.f.

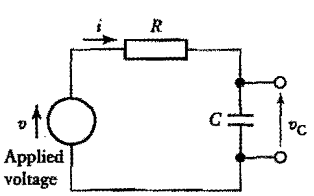


Figure 10.8 Resistor–capacitor system.

Now consider a simple electrical system consisting of a resistor and capacitor in series, as shown in Figure 10.8. Applying Kirchhoff 's second law to the circuit loop gives

$$v = v_R + v_C$$

where v_R is the potential difference across the resistor and v_C that across the capacitor. Since this is just a single loop, the current i through all the circuit elements will be the same. If the output from the circuit is the potential difference across the capacitor, v_C, then since $v_R = iR$ and $i = C(dv_C/dt)$,

$$v = RC\frac{dv_C}{dt} + v_C$$

This gives the relationship between the output v_C and the input v and is a first-order differential equation.

Figure 10.9 shows a resistor–inductor–capacitor system. If Kirchhoff's second law is applied to this circuit loop,

$$v = v_R + v_L + v_C$$

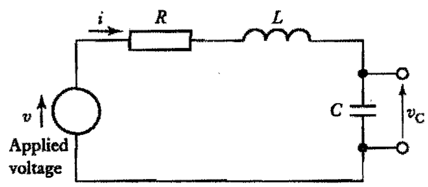


Figure 10.9
Resistor–inductor–capacitor system.

where v_R is the potential difference across the resistor, v_L that across the inductor and v_C that across the capacitor. Since there is just a single loop, the current i will be the same through all circuit elements. If the output from the circuit is the potential difference across the capacitor, v_C, then since $v_R = iR$ and $v_L = L(di/dt)$

$$v = iR + L\frac{di}{dt} + v_C$$

But $i = C(dv_C/dt)$ and so

$$\frac{di}{dt} = C\frac{d(dv_C/dt)}{dt} = C\frac{d^2v_C}{dt^2}$$

Hence

$$v = RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C$$

This is a second-order differential equation.

As a further illustration, consider the relationship between the output, the potential difference across the inductor of v_L, and the input v for the circuit shown in Figure 10.10. Applying Kirchhoff's second law to the circuit loop gives

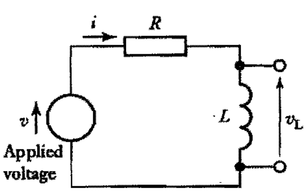


Figure 10.10
Resistor–inductor system.

$$v = v_R + v_L$$

where v_R is the potential difference across the resistor R and v_L that across the inductor. Since $v_R = iR$,

$$v = iR + v_L$$

Since

$$i = \frac{1}{L}\int v_L\,dt$$

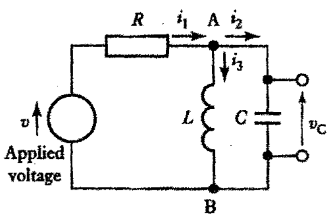


Figure 10.11 Resistor–capacitor–inductor system.

then the relationship between the input and output is

$$v = \frac{R}{L}\int v_L\,dt + v_L$$

As another example, consider the relationship between the output, the potential difference v_C across the capacitor, and the input v for the circuit shown in Figure 10.11. Applying Kirchhoff's law 1 to node A gives

$$i_1 = i_2 + i_3$$

But

$$i_1 = \frac{v - v_A}{R}$$

$$i_2 = \frac{1}{L}\int v_A\,dt$$

$$i_3 = C\frac{dv_A}{dt}$$

Hence

$$\frac{v - v_A}{R} = \frac{1}{L}\int v_A\,dt + C\frac{dv_A}{dt}$$

But $v_C = v_A$. Hence, with some rearrangement,

$$v = RC\frac{dv_C}{dt} + v_C + \frac{R}{L}\int v_C\,dt$$

10.3.2 Electrical and mechanical analogies

The building blocks for electrical and mechanical systems have many similarities (Figure 10.12). For example, the electrical resistor does not store energy but dissipates it, with the current i through the resistor being given by $i = v/R$, where R is a constant, and the power P dissipated by $P = v^2/R$. The mechanical analogue of the resistor is the dashpot. It also does not store energy but dissipates it, with the force F being related to the velocity v by $F = cv$, where c is a constant, and the power P dissipated by $P = cv^2$. Both these sets of equations have similar forms. Comparing them, and taking the current as being analogous to the force, then the potential difference is analogous

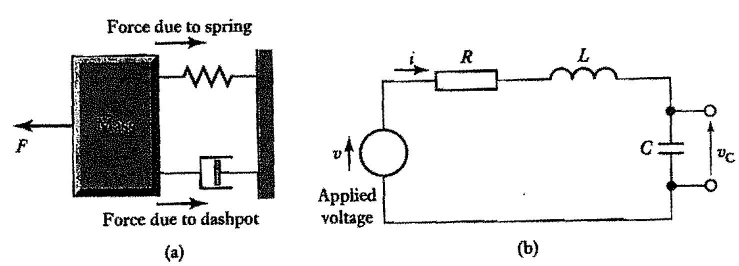


Figure 10.12 Analogous systems.

to the velocity and the dashpot constant c to the reciprocal of the resistance, i.e. $(1/R)$. These analogies between current and force, potential difference and velocity, hold for the other building blocks with the spring being analogous to inductance and mass to capacitance.

The mechanical system in Figure 10.1(a) and the electrical system in Figure 10.1(b) have input/output relationships described by similar differential equations:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \quad \text{and} \quad RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = v$$

The analogy between current and force is the one most often used. However, another set of analogies can be drawn between potential difference and force.

10.4

In fluid flow systems there are three basic building blocks which can be considered to be the equivalent of electrical resistance, capacitance and inductance. Fluid systems can be considered to fall into two categories: hydraulic, where the fluid is a liquid and is deemed to be incompressible; and pneumatic, where it is a gas which can be compressed and consequently shows a density change.

Hydraulic resistance is the resistance to flow which occurs as a result of a liquid flowing through valves or changes in a pipe diameter (Figure 10.13(a)). The relationship between the volume rate of flow of liquid q through the resistance element and the resulting pressure difference $(p_1 - p_2)$ is

$$p_1 - p_2 = Rq$$

where R is a constant called the hydraulic resistance. The bigger the resistance, the bigger the pressure difference for a given rate of flow. This equation, like that for the electrical resistance and Ohm's law, assumes a linear relationship. Such hydraulic linear resistances occur with orderly flow through capillary tubes and porous plugs but non-linear resistances occur with flow through sharp-edged orifices or if flow is turbulent.

Hydraulic capacitance is the term used to describe energy storage with a liquid where it is stored in the form of potential energy. A height of liquid in a container (Figure 10.13(b)), i.e. a so-called pressure head, is one form of

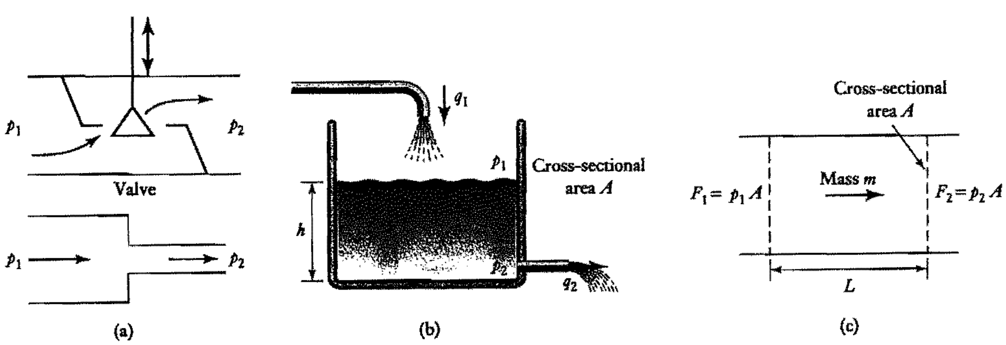


Figure 10.13 Hydraulic examples: (a) resistance, (b) capacitance, (c) inertance.

such a storage. For such a capacitance, the rate of change of volume V in the container, i.e. $\mathrm{d}V/\mathrm{d}t$, is equal to the difference between the volumetric rate at which liquid enters the container q_1 and the rate at which it leaves q_2,

$$q_1 - q_2 = \frac{\mathrm{d}V}{\mathrm{d}t}$$

But $V = Ah$, where A is the cross-sectional area of the container and h the height of liquid in it. Hence

$$q_1 - q_2 = \frac{\mathrm{d}(Ah)}{\mathrm{d}t} = A\frac{\mathrm{d}h}{\mathrm{d}t}$$

But the pressure difference between the input and output is p, where $p = h\rho g$ with ρ being the liquid density and g the acceleration due to gravity. Thus, if the liquid is assumed to be incompressible, i.e. its density does not change with pressure,

$$q_1 - q_2 = A\frac{\mathrm{d}(p/\rho g)}{\mathrm{d}t} = \frac{A}{\rho g}\frac{\mathrm{d}p}{\mathrm{d}t}$$

The hydraulic capacitance C is defined as being

$$C = \frac{A}{\rho g}$$

Thus

$$q_1 - q_2 = C\frac{\mathrm{d}p}{\mathrm{d}t}$$

Integration of this equation gives

$$p = \frac{1}{C}\int (q_1 - q_2)\,\mathrm{d}t$$

Hydraulic inertance is the equivalent of inductance in electrical systems or a spring in mechanical systems. To accelerate a fluid and so increase its velocity, a force is required. Consider a block of liquid of mass m (Figure 10.13(c)). The net force acting on the liquid is

$$F_1 - F_2 = p_1 A - p_2 A = (p_1 - p_2)A$$

where $(p_1 - p_2)$ is the pressure difference and A the cross-sectional area. This net force causes the mass to accelerate with an acceleration a, and so

$$(p_1 - p_2)A = ma$$

But a is the rate of change of velocity $\mathrm{d}v/\mathrm{d}t$, hence

$$(p_1 - p_2)A = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

But the mass of liquid concerned has a volume of AL, where L is the length of the block of liquid or the distance between the points in the liquid where the pressures p_1 and p_2 are measured. If the liquid has a density ρ then $m = AL\rho$ and so

$$(p_1 - p_2)A = AL\rho\frac{\mathrm{d}v}{\mathrm{d}t}$$

But the volume rate of flow $q = Av$, hence

$$(p_1 - p_2)A = L\rho\frac{dq}{dt}$$

$$p_1 - p_2 = I\frac{dq}{dt}$$

where the hydraulic inertance I is defined as

$$I = \frac{L\rho}{A}$$

With pneumatic systems the three basic building blocks are, as with hydraulic systems, resistance, capacitance and inertance. However, gases differ from liquids in being compressible, i.e. a change in pressure causes a change in volume and hence density. **Pneumatic resistance** R is defined in terms of the mass rate of flow dm/dt (note that this is often written as an m with a dot above it to indicate that the symbol refers to the mass rate of flow and not just the mass) and the pressure difference $(p_1 - p_2)$ as

$$p_1 - p_2 = R\frac{dm}{dt} = R\dot{m}$$

Pneumatic capacitance C is due to the compressibility of the gas, and is comparable with the way in which the compression of a spring stores energy. If there is a mass rate of flow dm_1/dt entering a container of volume V and a mass rate of flow of dm_2/dt leaving it, then the rate at which the mass in the container is changing is $(dm_1/dt - dm_2/dt)$. If the gas in the container has a density ρ then the rate of change of mass in the container is

$$\text{rate of change of mass in container} = \frac{d(\rho V)}{dt}$$

But, because a gas can be compressed, both ρ and V can vary with time. Hence

$$\text{rate of change of mass in container} = \rho\frac{dV}{dt} + V\frac{d\rho}{dt}$$

Since $(dV/dt) = (dV/dp)(dp/dt)$ and, for an ideal gas, $pV = mRT$ with consequently $p = (m/V)RT = \rho RT$ and $d\rho/dt = (1/RT)(dp/dt)$, then

$$\text{rate of change of mass in container} = \rho\frac{dV}{dp}\frac{dp}{dt} + \frac{V}{RT}\frac{dp}{dt}$$

where R is the gas constant and T the temperature, assumed to be constant, on the Kelvin scale. Thus

$$\frac{dm_1}{dt} - \frac{dm_2}{dt} = \left(\rho\frac{dV}{dp} + \frac{V}{RT}\right)\frac{dp}{dt}$$

The pneumatic capacitance due to the change in volume of the container C_1 is defined as

$$C_1 = \rho\frac{dV}{dp}$$

and the pneumatic capacitance due to the compressibility of the gas C_2 as

$$C_2 = \frac{V}{RT}$$

Hence

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} - \frac{\mathrm{d}m_2}{\mathrm{d}t} = (C_1 + C_2)\frac{\mathrm{d}p}{\mathrm{d}t}$$

or

$$p_1 - p_2 = \frac{1}{C_1 + C_2}\int(\dot{m}_1 - \dot{m}_2)\,\mathrm{d}t$$

Pneumatic inertance is due to the pressure drop necessary to accelerate a block of gas. According to Newton's second law, the net force is $ma = \mathrm{d}(mv)/\mathrm{d}t$. Since the force is provided by the pressure difference $(p_1 - p_2)$, then if A is the cross-sectional area of the block of gas being accelerated

$$(p_1 - p_2)A = \frac{\mathrm{d}(mv)}{\mathrm{d}t}$$

But m, the mass of the gas being accelerated, equals ρLA with ρ being the gas density and L the length of the block of gas being accelerated. And the volume rate of flow $q = Av$, where v is the velocity. Thus

$$mv = \rho LA\frac{q}{A} = \rho Lq$$

and so

$$(p_1 - p_2)A = L\frac{\mathrm{d}(\rho q)}{\mathrm{d}t}$$

But $\dot{m} = \rho q$ and so

$$p_1 - p_2 = \frac{L}{A}\frac{\mathrm{d}\dot{m}}{\mathrm{d}t}$$

$$p_1 - p_2 = I\frac{\mathrm{d}\dot{m}}{\mathrm{d}t}$$

with the pneumatic inertance I being $I = L/A$.

Table 10.3 shows the basic characteristics of the fluid building blocks, both hydraulic and pneumatic.

For hydraulics the volumetric rate of flow and for pneumatics the mass rate of flow are analogous to the electric current in an electrical system. For both hydraulics and pneumatics the pressure difference is analogous to the potential difference in electrical systems. Compare Table 10.3 with Table 10.2. Hydraulic and pneumatic inertance and capacitance are both energy storage elements; hydraulic and pneumatic resistance are both energy dissipaters.

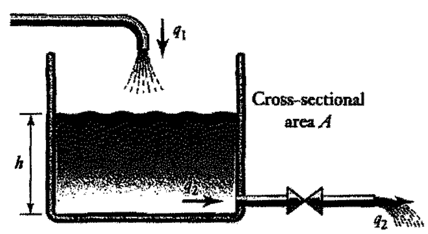
10.4.1 Building up a model for a fluid system

Figure 10.14 shows a simple hydraulic system, a liquid entering and leaving a container. Such a system can be considered to consist of a capacitor, the liquid in the container, with a resistor, the valve.

Table 10.3 Hydraulic and pneumatic building blocks

Building block	Describing equation	Energy stored or power dissipated
Hydraulic		
Inertance	$q = \frac{1}{L}\int (p_1 - p_2)\,\mathrm{d}t$ $p = L\frac{\mathrm{d}q}{\mathrm{d}t}$	$E = \frac{1}{2}Iq^2$
Capacitance	$q = C\frac{\mathrm{d}(p_1 - p_2)}{\mathrm{d}t}$	$E = \frac{1}{2}C(p_1 - p_2)^2$
Resistance	$q = \frac{p_1 - p_2}{R}$	$P = \frac{1}{R}(p_1 - p_2)^2$
Pneumatic		
Inertance	$\dot{m} = \frac{1}{L}\int (p_1 - p_2)\,\mathrm{d}t$	$E = \frac{1}{2}I\dot{m}^2$
Capacitance	$\dot{m} = C\frac{\mathrm{d}(p_1 - p_2)}{\mathrm{d}t}$	$E = \frac{1}{2}C(p_1 - p_2)^2$
Resistance	$\dot{m} = \frac{p_1 - p_2}{R}$	$P = \frac{1}{R}(p_1 - p_2)^2$

Figure 10.14 A fluid system.



Inertance can be neglected since flow rates change only very slowly. For the capacitor we can write

$$q_1 - q_2 = C\frac{\mathrm{d}p}{\mathrm{d}t}$$

The rate at which liquid leaves the container q_2 equals the rate at which it leaves the valve. Thus for the resistor

$$p_1 - p_2 = Rq_2$$

The pressure difference $(p_1 - p_2)$ is the pressure due to the height of liquid in the container and is thus $h\rho g$. Thus $q_2 = h\rho g/R$ and so substituting for q_2 in the first equation gives

$$q_1 - \frac{h\rho g}{R} = C\frac{\mathrm{d}(h\rho g)}{\mathrm{d}t}$$

and, since $C = A/\rho g$,

$$q_1 = A\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\rho g h}{R}$$

This equation describes how the height of liquid in the container depends on the rate of input of liquid into the container.

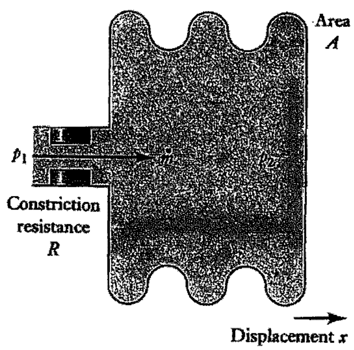


Figure 10.15 A pneumatic system.

A bellows is an example of a simple pneumatic system (Figure 10.15). Resistance is provided by a constriction which restricts the rate of flow of gas into the bellows and capacitance is provided by the bellows itself. Inertance can be neglected since the flow rate changes only slowly.

The mass flow rate into the bellows is given by

$$p_1 - p_2 = R\dot{m}$$

where p_1 is the pressure prior to the constriction and p_2 the pressure after the constriction, i.e. the pressure in the bellows. All the gas that flows into the bellows remains in the bellows, there being no exit from the bellows. The capacitance of the bellows is given by

$$\dot{m}_1 - \dot{m}_2 = (C_1 + C_2)\frac{dp_2}{dt}$$

The mass flow rate entering the bellows is given by the equation for the resistance and the mass leaving the bellows is zero. Thus

$$\frac{p_1 - p_2}{R} = (C_1 + C_2)\frac{dp_2}{dt}$$

Hence

$$p_1 = R(C_1 + C_2)\frac{dp_2}{dt} + p_2$$

This equation describes how the pressure in the bellows p_2 varies with time when there is an input of a pressure p_1.

The bellows expands or contracts as a result of pressure changes inside it. Bellows are just a form of spring and so we can write $F = kx$ for the relationship between the force F causing an expansion or contraction and the resulting displacement x, where k is the spring constant for the bellows. But the force F depends on the pressure p_2, with $p_2 = F/A$ where A is the cross-sectional area of the bellows. Thus $p_2A = F = kx$. Hence substituting for p_2 in the above equation gives

$$p_1 = R(C_1 + C_2)\frac{k}{A}\frac{dx}{dt} + \frac{k}{A}x$$

This equation, a first-order differential equation, describes how the extension or contraction x of the bellows changes with time when there is an input of a pressure p_1. The pneumatic capacitance due to the change in volume of the container C_1 is $\rho dV/dp_2$ and since $V = Ax$, C_1 is $\rho A\, dx/dp_2$. But for the bellows $p_2A = kx$, thus

$$C_1 = \rho A\frac{dx}{d(kx/A)} = \frac{\rho A^2}{k}$$

C_2, the pneumatic capacitance due to the compressibility of the air, is $V/RT = Ax/RT$.

The following illustrates how, for the hydraulic system shown in Figure 10.16, relationships can be derived which describe how the heights of the liquids in the two containers will change with time. With this model inertance is neglected.

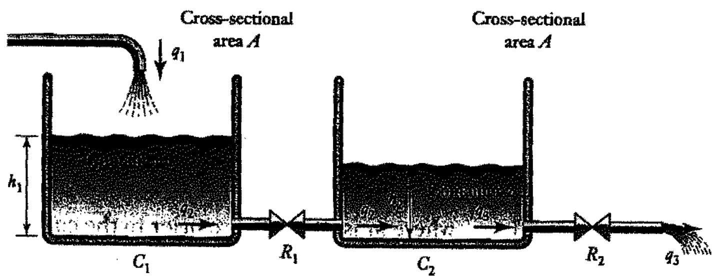


Figure 10.16 A fluid system.

Container 1 is a capacitor and thus

$$q_1 - q_2 = C_1 \frac{dp}{dt}$$

where $p = h_1 \rho g$ and $C_1 = A_1/\rho g$ and so

$$q_1 - q_2 = A_1 \frac{dh_1}{dt}$$

The rate at which liquid leaves the container q_2 equals the rate at which it leaves the valve R_1. Thus for the resistor,

$$p_1 - p_2 = R_1 q_2$$

The pressures are $h_1 \rho g$ and $h_2 \rho g$. Thus

$$(h_1 - h_2)\rho g = R_1 q_2$$

Using the value of q_2 given by this equation and substituting it into the earlier equation gives

$$q_1 - \frac{(h_1 - h_2)\rho g}{R_1} = A_1 \frac{dh_1}{dt}$$

This equation describes how the height of the liquid in container 1 depends on the input rate of flow.

For container 2 a similar set of equations can be derived. Thus for the capacitor C_2,

$$q_2 - q_3 = C_2 \frac{dp}{dt}$$

where $p = h_2 \rho g$ and $C_2 = A_2/\rho g$ and so

$$q_2 - q_3 = A_2 \frac{dh_2}{dt}$$

The rate at which liquid leaves the container q_3 equals the rate at which it leaves the valve R_2. Thus for the resistor,

$$p_2 - 0 = R_2 q_3$$

This assumes that the liquid exits into the atmosphere. Thus, using the value of q_3 given by this equation and substituting it into the earlier equation gives

$$q_2 - \frac{h_2 \rho g}{R_2} = A_2 \frac{dh_2}{dt}$$

Substituting for q_2 in this equation using the value given by the equation derived for the first container gives

$$\frac{(h_1 - h_2)\rho g}{R_1} - \frac{h_2 \rho g}{R_2} = A_2 \frac{\mathrm{d}h_2}{\mathrm{d}t}$$

This equation describes how the height of liquid in container 2 changes.

10.5

There are only two basic building blocks for thermal systems: resistance and capacitance. There is a net flow of heat between two points if there is a temperature difference between them. The electrical equivalent of this is that there is only a net current i between two points if there is a potential difference v between them, the relationship between the current and potential difference being $i = v/R$, where R is the electrical resistance between the points. A similar relationship can be used to define **thermal resistance** R. If q is the rate of flow of heat and $(T_1 - T_2)$ the temperature difference, then

$$q = \frac{T_2 - T_1}{R}$$

The value of the resistance depends on the mode of heat transfer. In the case of conduction through a solid, for unidirectional conduction

$$q = Ak\frac{T_1 - T_2}{L}$$

where A is the cross-sectional area of the material through which the heat is being conducted and L the length of material between the points at which the temperatures are T_1 and T_2; k is the thermal conductivity. Hence, with this mode of heat transfer,

$$R = \frac{L}{Ak}$$

When the mode of heat transfer is convection, as with liquids and gases, then

$$q = Ah(T_2 - T_1)$$

where A is the surface area across which there is the temperature difference and h is the coefficient of heat transfer. Thus, with this mode of heat transfer,

$$R = \frac{1}{Ah}$$

Thermal capacitance is a measure of the store of internal energy in a system. Thus, if the rate of flow of heat into a system is q_1 and the rate of flow out is q_2, then

$$\text{rate of change of internal energy} = q_1 - q_2$$

An increase in internal energy means an increase in temperature. Since

$$\text{internal energy change} = mc \times \text{change in temperature}$$

where m is the mass and c the specific heat capacity, then

$$\text{rate of change of internal energy} = mc \times \text{rate of change of temperature}$$

Thus

$$q_1 - q_2 = mc\frac{\mathrm{d}T}{\mathrm{d}t}$$

where $\mathrm{d}T/\mathrm{d}t$ is the rate of change of temperature. This equation can be written as

$$q_1 - q_2 = C\frac{\mathrm{d}T}{\mathrm{d}t}$$

where C is the thermal capacitance and so $C = mc$. Table 10.4 gives a summary of the thermal building blocks.

Table 10.4 Thermal building blocks.

Building block	Describing equation	Energy stored
Capacitance	$q_1 - q_2 = C\frac{\mathrm{d}T}{\mathrm{d}t}$	$E = CT$
Resistance	$q = \frac{T_1 - T_2}{R}$	

Building up a model for a thermal system

Consider a thermometer at temperature T which has just been inserted into a liquid at temperature T_L (Figure 10.17).

If the thermal resistance to heat flow from the liquid to the thermometer is R, then

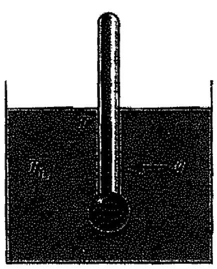


Figure 10.17 A thermal system.

$$q = \frac{T_L - T}{R}$$

where q is the net rate of heat flow from liquid to thermometer. The thermal capacitance C of the thermometer is given by the equation

$$q_1 - q_2 = C\frac{\mathrm{d}T}{\mathrm{d}t}$$

Since there is only a net flow of heat from the liquid to the thermometer, $q_1 = q$ and $q_2 = 0$. Thus

$$q = C\frac{\mathrm{d}T}{\mathrm{d}t}$$

Substituting this value of q in the earlier equation gives

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{T_L - T}{R}$$

Rearranging this equation gives

$$RC\frac{\mathrm{d}T}{\mathrm{d}t} + T = T_L$$

This equation, a first-order differential equation, describes how the temperature

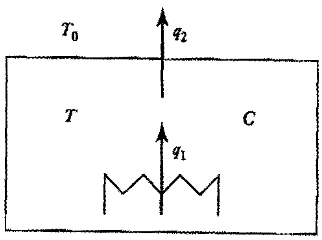


Figure 10.18 Thermal system.

indicated by the thermometer T will vary with time when the thermometer is inserted into a hot liquid.

In the above thermal system the parameters have been considered to be lumped. This means, for example, that there has been assumed to be just one temperature for the thermometer and just one for the liquid, i.e. the temperatures are only functions of time and not position within a body.

To illustrate the above consider Figure 10.18 which shows a thermal system consisting of an electric fire in a room. The fire emits heat at the rate q_1 and the room loses heat at the rate q_2. Assuming that the air in the room is at a uniform temperature T and that there is no heat storage in the walls of the room, derive an equation describing how the room temperature will change with time.

If the air in the room has a thermal capacity C then

$$q_1 - q_2 = C\frac{\mathrm{d}T}{\mathrm{d}t}$$

If the temperature inside the room is T and that outside the room T_0 then

$$q_2 = \frac{T - T_0}{R}$$

where R is the resistivity of the walls. Substituting for q_2 gives

$$q_1 - \frac{T - T_0}{R} = C\frac{\mathrm{d}T}{\mathrm{d}t}$$

Hence

$$RC\frac{\mathrm{d}T}{\mathrm{d}t} + T = Rq_1 + T_0$$

Summary

A **mathematical model** of a system is a description of it in terms of equations relating inputs and outputs so that outputs can be predicted from inputs.

Mechanical systems can be considered to be made up from masses, springs and dashpots, or moments of inertia, springs and rotational dampers if rotational. Electrical systems can be considered to be made up from resistors, capacitors and inductors, hydraulic and pneumatic systems from resistance, capacitance and inertance, and thermal systems from resistance and capacitance.

There are many elements in mechanical, electrical, fluid and thermal systems which have similar behaviours. Thus, for example, mass in mechanical systems has similar properties to capacitance in electrical systems, capacitance in fluid systems and capacitance in thermal systems. Table 10.5 shows a comparison of the elements in each of these systems and their defining equations.

Table 10.5 System elements.

	Mechanical (translational)	Mechanical (rotational)	Electrical	Fluid (hydraulic)	Thermal
Element	Mass	Moment of inertia	Capacitor	Capacitor	Capacitor
Equation	$F = m\dfrac{d^2x}{dt^2}$	$T = I\dfrac{d^2\theta}{dt^2}$			
	$F = m\dfrac{dv}{dt}$	$T = I\dfrac{d\omega}{dt}$	$i = C\dfrac{dv}{dt}$	$q = C\dfrac{d(p_1 - p_2)}{dt}$	$q_1 - q_2 = C\dfrac{dT}{dt}$
Energy	$E = \dfrac{1}{2}mv^2$	$E = \dfrac{1}{2}I\omega^2$	$E = \dfrac{1}{2}Cv^2$	$E = \dfrac{1}{2}C(p_1 - p_2)^2$	$E = CT$
Element	Spring	Spring	Inductor	Inertance	None
Equation	$F = kx$	$T = k\theta$	$v = L\dfrac{di}{dt}$	$p = L\dfrac{dq}{dt}$	
Energy	$E = \dfrac{1}{2}\dfrac{F^2}{k}$	$E = \dfrac{1}{2}\dfrac{T^2}{k}$	$E = \dfrac{1}{2}Li^2$	$E = \dfrac{1}{2}Iq^2$	
Element	Dashpot	Rotational damper	Resistor	Resistance	Resistance
Equation	$F = c\dfrac{dx}{dt} = cv$	$T = c\dfrac{d\theta}{dt} = c\omega$	$i = \dfrac{v}{R}$	$q = \dfrac{p_1 - p_2}{R}$	$q = \dfrac{T_1 - T_2}{R}$
Power	$P = cv^2$	$P = c\omega^2$	$P = \dfrac{v^2}{R}$	$P = \dfrac{1}{R}(p_1 - p_2)^2$	

Problems

10.1 Derive an equation relating the input, force F, with the output, displacement x, for the systems described by Figure 10.19.

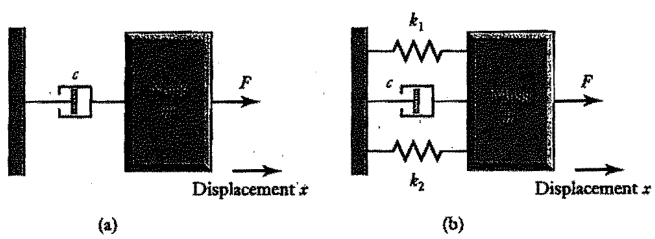


Figure 10.19 Problem 10.1.

10.2 Propose a model for the metal wheel of a railway carriage running on a metal track.

10.3 Derive an equation relating the input angular displacement θ_i with the output angular displacement θ_o for the rotational system shown in Figure 10.20.

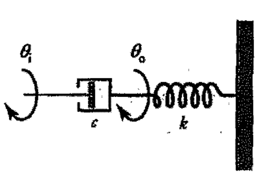


Figure 10.20 Problem 10.3.

10.4 Propose a model for a stepped shaft (i.e. a shaft where there is a step change in diameter) used to rotate a mass and derive an equation relating the input torque and the angular rotation. You may neglect damping.

10.5 Derive the relationship between the output, the potential difference across the resistor R of v_R, and the input v for the circuit shown in Figure 10.21 which has a resistor in series with a capacitor.

10.6 Derive the relationship between the output, the potential difference across the resistor R of v_R, and the input v for the series LCR circuit shown in Figure 10.22.

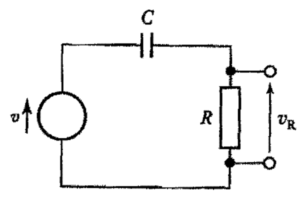


Figure 10.21 Problem 10.5.

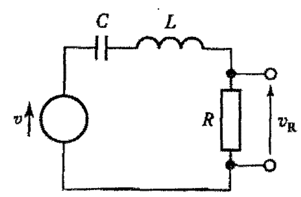


Figure 10.22 Problem 10.6.

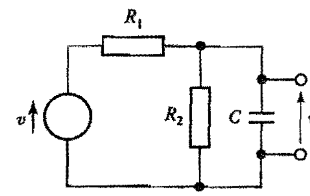
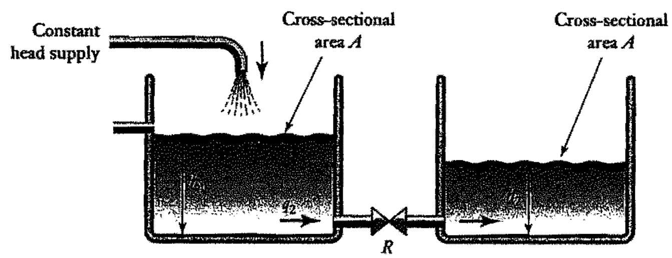


Figure 10.23 Problem 10.7.

10.7 Derive the relationship between the output, the potential difference across the capacitor C of v_C, and the input v for the circuit shown in Figure 10.23.

10.8 Derive the relationship between the height h_2 and time for the hydraulic system shown in Figure 10.24. Neglect inertance.

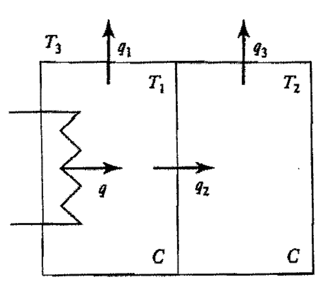
Figure 10.24
Problem 10.8.



10.9 A hot object, capacitance C and temperature T, cools in a large room at temperature T_r. If the thermal system has a resistance R, derive an equation describing how the temperature of the hot object changes with time and give an electrical analogue of the system.

10.10 Figure 10.25 shows a thermal system involving two compartments, with one containing a heater. If the temperature of the compartment containing the heater is T_1, the temperature of the other compartment T_2 and the temperature surrounding the compartments T_3, develop equations describing how

Figure 10.25
Problem 10.10.



the temperatures T_1 and T_2 will vary with time. All the walls of the containers have the same resistance and negligible capacitance. The two containers have the same capacitance C.

10.11 Derive the differential equation relating the pressure input p to a diaphragm actuator (as in Figure 7.23) to the displacement x of the stem.

10.12 Derive the differential equation for a motor driving a load through a gear system (Figure 10.26) which relates the angular displacement of the load with time.

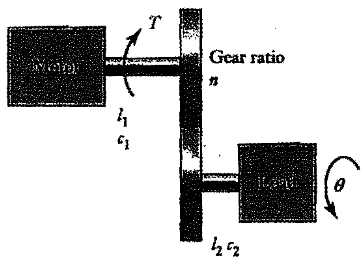


Figure 10.26
Problem 10.12.